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"BEST POSSIBLE" UPPER AND LOWER BOUNDS FOR THE ZEROS OF THE BESSEL FUNCTION $J_\nu(x)$

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Abstract. Let $j_{\nu,k}$ denote the k-th positive zero of the Bessel function $J_\nu(x)$. In this paper, we prove that for $\nu > 0$ and $k = 1, 2, 3, \ldots,$

$$\nu - \frac{a_k}{2^{1/3}}\nu^{1/3} < j_{\nu,k} < \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} + \frac{3}{20}a_k^2\frac{2^{1/3}}{\nu^{1/3}}.$$

These bounds coincide with the first few terms of the well-known asymptotic expansion

$$j_{\nu,k} \sim \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} + \frac{3}{20}a_k^2\frac{2^{1/3}}{\nu^{1/3}} + \cdots$$

as $\nu \to \infty$, k being fixed, where a_k is the k-th negative zero of the Airy function $\mathrm{Ai}(x)$, and so are "best possible".

1. Introduction

The k-th positive zero $j_{\nu,k}$ of the Bessel function $J_\nu(x)$ has the asymptotic expansion

$$j_{\nu,k} \sim \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} + \frac{3}{20}a_k^2\frac{2^{1/3}}{\nu^{1/3}} + \cdots \tag{1.1}$$

as $\nu \to \infty$, k being fixed, where a_k is the k-th negative zero of the Airy function $\mathrm{Ai}(x)$. Recently, Lorch conjectured that the sum of the first two terms in the expansion gives a lower bound for $j_{\nu,k}$, and that the sum of the first three terms gives an upper bound for $j_{\nu,k}$.

This result has been established by Lorch and Uberti [9] in the cases when $0 < \nu \le 10$ and $k = 1, 2, 3$, and by Lang and Wong [7] in the cases when $10 \le \nu < \infty$ and $k = 1, 2$. The purpose of this paper is to show that Lorch's conjecture is true for all $\nu > 0$ and $k = 1, 2, 3, \cdots$; that is, we shall prove

$$\nu - \frac{a_k}{2^{1/3}}\nu^{1/3} < j_{\nu,k} < \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} + \frac{3}{20}a_k^2\frac{2^{1/3}}{\nu^{1/3}}. \tag{1.2}$$

Zeros of Bessel functions occur frequently in eigenvalue problems associated with the Laplacian with Dirichlet boundary conditions; see, for example, [1] and [13]. The inequalities in (1.2) provide sharp bounds for these eigenvalues. For other recent results on inequalities for $j_{\nu,k}$, see [3] and [8].

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During the process of writing up our result, we were informed by Lorch of a recent paper of Breen [2], in which it is proved that for $\nu \geq \frac{1}{2}$ and $k = 1, 2, 3, \cdots$,

$$\nu - \frac{a_{k-1}}{2^{1/3}}\nu^{1/3} < j_{\nu,k} \tag{1.3a}$$

and

$$j_{\nu,k} < \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} + \frac{a_k^2}{2^{2/3}}\nu^{-1/3}\left(1 + \frac{a_k}{2^{1/3}}\right)^{-1}. \tag{1.3b}$$

Breen noticed the close resemblance between the bounds in (1.3a) and (1.3b) and the first three terms in the asymptotic expansion (1.1). However, it is evident that the bounds in (1.2) are an even closer (in fact, exact) match to the the first three terms in expansion (1.1).

Our method is based essentially on the argument used by Lang and Wong in [7], except that here we need to extend the range of validity of a crucial result (Lemma 1) and to deal with the dependence of $j_{\nu,k}$ on the variable k.

The presentation of the paper is arranged as follows. In § 2, we recall a uniform asymptotic approximation for the Bessel function $J_\nu(x)$. In § 3, we present a brief outline of the argument used in [7]. In §§ 4 and 5, we give some estimates related to the Airy function and its zeros. Two crucial lemmas are proved in § 6. The inequalities in (1.2) are established in §§ 7 and 8.

2. A uniform approximation for $J_\nu(\nu x)$

The Bessel function $J_\nu(x)$ has the well-known uniform asymptotic approximation

$$J_\nu(\nu x) = \frac{1}{1+\delta_3}\frac{\varphi(\zeta)}{\nu^{1/3}}\left\{\mathrm{Ai}(\nu^{2/3}\zeta)\left[1 + \frac{A_1(\zeta)}{\nu^2}\right] + \frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{4/3}}B_0(\zeta) + \varepsilon_3(\nu,\zeta)\right\}, \tag{2.1}$$

valid for $\nu > 0$ and $x > 0$, where ζ and x are related in a one-to-one manner by the equations

$$\frac{2}{3}\zeta^{3/2} = \ln\frac{1+(1-x^2)^{1/2}}{x} - (1-x^2)^{1/2}, \qquad 0 < x \leq 1, \tag{2.2}$$

$$\frac{2}{3}(-\zeta)^{3/2} = (x^2-1)^{1/2} - \sec^{-1} x, \qquad x \geq 1, \tag{2.3}$$

and where

$$\varphi(\zeta) = \left(\frac{4\zeta}{1-x^2}\right)^{1/4}; \tag{2.4}$$

see [12, Chapter 11] and also [10, 11]. The coefficients are analytic functions in a region containing the real axis and can be given explicitly. The error term $\varepsilon_3(\nu,\zeta)$ satisfies a realistic numerical bound. To state the result, we recall from [12, p.395] the *modulus function* $M(x)$ and the *weight function* $E(x)$, defined as follows.

Let $x = c$ be the negative root of the equation $\mathrm{Ai}(x) = \mathrm{Bi}(x)$ with the smallest absolute value. Numerical calculation gives $c = -0.36605$. The function $E(x)$ and

$M(x)$ are defined by

$$E(x) = \left\{ \mathrm{Bi}(x)/\,\mathrm{Ai}(x) \right\}^{1/2}, \qquad c \le x < \infty, \tag{2.5}$$

$$E(x) = 1, \qquad -\infty < x \le c; \qquad E^{-1}(x) = 1/E(x), \tag{2.6}$$

$$M(x) = \left\{ E^2(x)\,\mathrm{Ai}^2(x) + E^{-2}(x)\,\mathrm{Bi}^2(x) \right\}^{1/2}. \tag{2.7}$$

The following constants also occur in our later calculations:

$$\lambda = \sup_{(-\infty,\infty)} \left\{ \pi |x|^{1/2} M^2(x) \right\} = 1.04\ldots\,, \tag{2.8}$$

$$\mu = \sup_{(-\infty,c)} \left\{ \pi |x|^{1/2} M^2(x) \right\} = 1. \tag{2.9}$$

The error term satisfies

$$\left|\varepsilon_3(\nu,\zeta)\right| \le \frac{2M(\nu^{2/3}\zeta)}{E(\nu^{2/3}\zeta)} \exp\left\{ \frac{2\lambda}{\nu} \mathcal{V}_{\zeta,\infty}(|\zeta|^{1/2} B_0(\zeta)) \right\} \frac{\mathcal{V}_{\zeta,\infty}(|\zeta|^{1/2} B_1(\zeta))}{\nu^3}, \tag{2.10}$$

where $\mathcal{V}_{a,b}(f)$ denotes the total variation of a function f on an interval (a,b). Numerical calculation gives

$$\mathcal{V}_{-\infty,\infty}\left\{ |\zeta|^{1/2} B_0(\zeta) \right\} = 0.1051, \tag{2.11}$$

$$\mathcal{V}_{-\infty,\infty}\left\{ |\zeta|^{1/2} B_1(\zeta) \right\} = 0.0066135; \tag{2.12}$$

see [10, p.7] and [7]. (Analytical proofs of these results can be provided upon request.) In order to use the approximation in (2.1), we also need an estimate for the term δ_3, which is given by

$$\left|\delta_3\right| \le 2e^{\nu_0/\nu} \nu^{-3} \mathcal{V}_{-\infty,\infty}\big(|\zeta|^{1/2} B_1(\zeta)\big), \tag{2.13}$$

where

$$\nu_0 = 2\lambda \mathcal{V}_{-\infty,\infty}\big(|\zeta|^{1/2} B_0(\zeta)\big) = 0.22\,. \tag{2.14}$$

We shall rewrite (2.1) in a different form. By Taylor's theorem

(2.15)
$$\mathrm{Ai}\left\{ v^{2/3}\zeta + \frac{B_0(\zeta)}{\nu^{4/3}(1 + A_1(\zeta)/\nu^2)} \right\} = \mathrm{Ai}(\nu^{2/3}\zeta) + \mathrm{Ai}'(\nu^{2/3}\zeta) \frac{B_0(\zeta)}{\nu^{4/3}(1 + A_1(\zeta)/\nu^2)} + \frac{\mathrm{Ai}''(\theta_1)}{2!} \frac{B_0^2(\zeta)}{\nu^{8/3}(1 + A_1(\zeta)/\nu^2)^2},$$

where θ_1 lies between $\nu^{2/3}\zeta$ and

$$\nu^{2/3}\zeta + \frac{B_0(\zeta)}{\left[\nu^{4/3}(1 + A_1(\zeta)/\nu^2)\right]}.$$

Numerical values of $A_1(\zeta)$ given in [10, Table 2] indicate that $A_1(\zeta)$ is a decreasing function in $(-\infty, 0)$, and that

$$-\frac{1}{225} = A_1(0) \le A_1(\zeta) < 0, \qquad \zeta \in (-\infty, 0]. \tag{2.16}$$

This can be confirmed analytically by an argument analogous to that given in [4, Appendix I]. Thus, as long as $\zeta < 0$ and $\nu^2 > 1/225$, then $1 + A_1(\zeta)\nu^{-2}$ is positive. Since $\varphi(\zeta)$ does not vanish in $(-\infty, \infty)$, equation (2.1) can be written as

$$(2.17)\quad \frac{(1+\delta_3)\nu^{1/3}}{\varphi(\zeta)(1+A_1(\zeta)/\nu^2)} J_\nu(\nu x) = \mathrm{Ai}\left\{\nu^{2/3}\zeta + \frac{B_0(\zeta)}{\nu^{4/3}(1+A_1(\zeta)/\nu^2)}\right\} + \varepsilon(\nu,\zeta),$$

where

$$(2.18)\quad \varepsilon(\nu,\zeta) = \frac{\varepsilon_3(\nu,\zeta)}{1+A_1(\zeta)/\nu^2} - \frac{\mathrm{Ai}''(\theta_1)}{2!}\frac{B_0^2(\zeta)}{\nu^{8/3}(1+A_1(\zeta)/\nu^2)^2}.$$

3. Argument of Lang and Wong

As in [7], we shall make use of the following result of Hethcote [4, p.14].

Theorem. *In the interval $[a_k-\rho, a_k+\rho']$ where a_k is the k-th negative zero of $\mathrm{Ai}(t)$ and ρ, ρ' are small enough so that $m = \min|\mathrm{Ai}'(t)| > 0$, suppose $f(t) = \mathrm{Ai}(t)+\varepsilon(t)$, $f(t)$ is continuous, and $E = \max|\varepsilon(t)| < \min\{|\mathrm{Ai}(a_k-\rho)|,\ |\mathrm{Ai}(a_k+\rho')|\}$. Then there exists a zero t_k of $f(t)$ in the interval such that $|t_k - a_k| \le E/m$.*

We apply this theorem to (2.17) with

$$(3.1)\quad t = \nu^{2/3}\zeta + \frac{B_0(\zeta)}{\nu^{4/3}(1+A_1(\zeta)/\nu^2)}$$

as the independent variable,

$$(3.2)\quad f(t) = \frac{(1+\delta_3)\nu^{1/3}}{\varphi(\zeta)(1+A_1(\zeta)/\nu^2)} J_\nu(\nu x),$$

and $\varepsilon(t) = \varepsilon(\nu,\zeta)$ given in (2.18). The result is that for suitably chosen ρ and ρ', there exists t_k such that $f(t_k) = 0$. Note that $B_0(\zeta)$ is increasing and $A_1(\zeta)$ is decreasing in $(-\infty, 0)$; cf. the last paragraph in § 2. Hence, it is easily seen from (3.1) that $dt/d\zeta$ is positive and t is an increasing function of ζ in $(-\infty, 0)$. Also note that the factor on the left-hand side of $J_\nu(\nu x)$ does not vanish for $\zeta < 0$ and $\nu^2 > 1/225$. Therefore, corresponding to t_k, there exist ζ_k and, by (2.3), x_k such that $J_\nu(\nu x_k) = 0$ and

$$(3.3)\quad \left|\nu^{2/3}\zeta_k + \frac{B_0(\zeta_k)}{\nu^{4/3}(1+A_1(\zeta_k)/\nu^2)} - a_k\right| \le \frac{E}{m},$$

where E and m are as given in the theorem.

Let

$$(3.4)\quad \nu^{2/3}\zeta_k + \frac{B_0(\zeta_k)}{\nu^{4/3}(1+A_1(\zeta_k)/\nu^2)} = a_k + \eta_k$$

and

$$(3.5)\quad \delta_k = -\frac{B_0(\zeta_k)}{(1+A_1(\zeta_k)/\nu^2)} + \eta_k\nu^{4/3},$$

so that

$$(3.6)\quad \zeta_k = (a_k + \delta_k\nu^{-4/3})\nu^{-2/3}.$$

Putting this in the Maclaurin expansion [12, p.421]

$$x(\zeta) = 1 - 2^{-1/3}\zeta + \frac{3}{10}2^{-2/3}\zeta^2 + \frac{1}{700}\zeta^3 + \cdots,$$

we obtain

$$
\begin{aligned}
x_k = x_k(\zeta_k) &= 1 - 2^{-1/3} a_k \nu^{-2/3} + \frac{3}{20} 2^{1/3} a_k^2 \nu^{-4/3} - 2^{-1/3} \delta_k \nu^{-2} \\
&\quad + \frac{3}{20} 2^{1/3} (2a_k \delta_k \nu^{-4/3} + \delta_k^2 \nu^{-8/3}) \nu^{-4/3} + \frac{x'''(\theta)}{3!} (a_k + \delta_k \nu^{-4/3})^3 \nu^{-2},
\end{aligned}
\tag{3.7}
$$

where $\zeta_k < \theta < 0$. Comparing with (1.1), it is evident that $\nu x_k = j_{\nu,k}$. Hence, we may write

$$
j_{\nu,k} = \nu - \frac{a_k}{2^{1/3}} \nu^{1/3} + \frac{3}{20} a_k^2 \frac{2^{1/3}}{\nu^{1/3}} + \beta \nu^{-1}
\tag{3.8}
$$

with

$$
\beta = -2^{-1/3} \delta_k + \frac{3}{20} 2^{1/3} \delta_k (2a_k + \delta_k \nu^{-4/3}) \nu^{-2/3} + \frac{x'''(\theta)}{6} (a_k + \delta_k \nu^{-4/3})^3 .
\tag{3.9}
$$

If we can show that β is negative when $\nu \geq 2$ and $k \geq 3$, then we will have established the upper bound in (1.2) under these conditions.

The above argument is essentially the one used by Lang and Wong in [7] to prove (1.2) in the case when $k = 1, 2$ and $\nu \geq 10$.

4. The choices for ρ and ρ'

First, we must make sure that in the interval $[a_k - \rho,\ a_k + \rho']$ we have $|\operatorname{Ai}'(t)| > 0$. Therefore, we choose ρ and ρ' so that

$$
a'_{k+1} < a_k - \rho < a_k < a_k + \rho' < a'_k .
\tag{4.1}
$$

Next, we want to make sure that the error term (2.18) in the asymptotic formula (2.17) satisfies

$$
\max |\varepsilon(\nu, \zeta)| < \min\Big(|\operatorname{Ai}(a_k - \rho)|,\ |\operatorname{Ai}(a_k + \rho')| \Big).
\tag{4.2}
$$

From [5], we have

$$
a_k \leq - \left[\frac{3\pi}{8} (4k - 1) \right]^{2/3}, \qquad k \geq 1.
\tag{4.3}
$$

Furthermore, in [4] and [14], it has been shown that

$$
a_k = - \left[\frac{3\pi}{8} (4k - 1) \right]^{2/3} (1 + \sigma_k),
\tag{4.4}
$$

where

$$
|\sigma_k| \leq 0.130 \left[\frac{3\pi}{8} (4k - 1.051) \right]^{-2}
\tag{4.5}
$$

for $k \geq 1$, and

$$
a'_k = - \left[\frac{3\pi}{8} (4k - 3) \right]^{2/3} (1 + \tau_k),
\tag{4.6}
$$

where

$$
|\tau_k| \leq 0.165 \left[\frac{3\pi}{8} (4k - 3.0382) \right]^{-2}
\tag{4.7}
$$

for $k \geq 2$. The inequalities in (4.1) suggest that we should choose ρ and ρ' so that

(4.8)
$$-\left[\frac{3\pi}{8}(4k-1)\right]^{2/3} < a_k + \rho' < -\left[\frac{3\pi}{8}(4k-3)\right]^{2/3}\left\{1+\frac{0.165}{[\frac{3\pi}{8}(4k-3.0382)]^2}\right\}$$

and

(4.9)
$$\begin{aligned} -\left[\frac{3\pi}{8}(4k+1)\right]^{2/3}\left\{1-\frac{0.165}{[\frac{3\pi}{8}(4k+0.9618)]^2}\right\} &< a_k - \rho \\ < -\left[\frac{3\pi}{8}(4k-1)\right]^{2/3}\left\{1+\frac{0.130}{[\frac{3\pi}{8}(4k-1.051)]^2}\right\}&. \end{aligned}$$

Put

(4.10)
$$a_{k,0} \equiv -\left[\frac{3\pi}{8}(4k-1)\right]^{2/3},$$

and note that

$$(4k-3)^{2/3} = (4k-1)^{2/3}\left(1-\frac{2}{4k-1}\right)^{2/3} < (4k-1)^{2/3}\left(1-\frac{4}{3}(4k-1)^{-1}\right)$$

and

$$\frac{1}{(4k-3.0382)^2} < \frac{0.137}{4k-1}, \qquad \text{for } k \geq 3.$$

Thus the right-hand side of (4.8) is greater than

$$a_{k,0}\left[1-\frac{4}{3(4k-1)}\right]\left[1+\frac{0.0163}{4k-1}\right] > a_{k,0}\left(1-\frac{1.3}{4k-1}\right)$$

for $k \geq 3$. For ρ' to satisfy (4.8), we choose it so that

(4.11)
$$a_{k,0} < a_k + \rho' < a_{k,0}\left(1-\frac{1.3}{4k-1}\right).$$

Using a similar argument, it can be shown that the right-hand side of (4.9) is greater than

$$a_{k,0}\left(1+\frac{0.009}{4k-1}\right)$$

and the left-hand side of (4.9) is less than

$$a_{k,0}\left(1+\frac{1.28}{4k-1}\right)$$

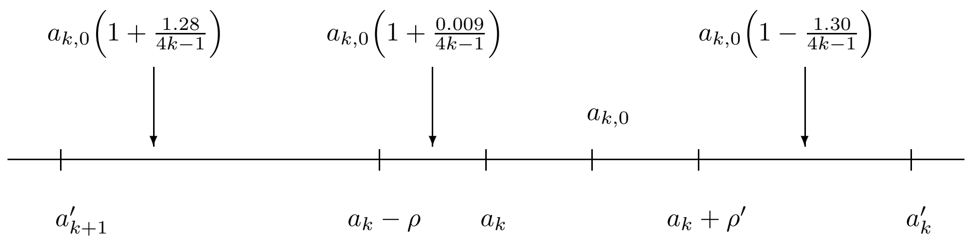
for $k \geq 3$. For ρ to satisfy (4.9), we choose it so that

(4.12)
$$a_{k,0}\left(1+\frac{1.28}{4k-1}\right) < a_k - \rho < a_{k,0}\left(1+\frac{0.009}{4k-1}\right).$$

On account of (4.11) and (4.12), we choose ρ and ρ' so that

(4.13)
$$\begin{aligned} a_k + \rho' &= a_{k,0}\left(1-\frac{0.01}{4k-1}\right), \\ a_k - \rho &= a_{k,0}\left(1+\frac{0.01}{4k-1}\right); \end{aligned}$$

see the figure below.



We now will show that with the choice of ρ and ρ' as given in (4.13), the estimate in (4.2) holds. By (3.1), if t lies in the interval $(a_k-\rho,\ a_k+\rho')$, then ζ satisfies

$$a_k-\rho-\frac{B_0(\zeta)}{\nu^{4/3}(1+A_1(\zeta)/\nu^2)}<\nu^{2/3}\zeta<a_k+\rho'-\frac{B_0(\zeta)}{\nu^{4/3}(1+A_1(\zeta)/\nu^2)}. \tag{4.14}$$

Since $B_0(\zeta)>0$ and $-1/225<A_1(\zeta)<0$ (see (2.16)), we have

$$\nu^{2/3}\zeta<a_k+\rho'<c=-0.36605, \qquad k=1,2,\cdots, \tag{4.15}$$

as long as $\nu^2>1/225$. A combination of (2.6), (2.8)–(2.10) then gives

$$|\varepsilon_3(\nu,\zeta)|\le\frac{2}{\sqrt{\pi}|\nu^{2/3}\zeta|^{1/4}}\exp\left\{\frac{2.08}{\nu}\mathcal{V}_{-\infty,\infty}(|\zeta|^{1/2}B_0)\right\}\frac{\mathcal{V}_{-\infty,\infty}(|\zeta|^{1/2}B_1)}{\nu^3},$$

which in turn yields

$$|\varepsilon_3(\nu,\zeta)|\le\frac{2}{\sqrt{\pi}|\nu^{2/3}\zeta|^{1/4}}e^{0.22/\nu}\frac{0.00662}{\nu^3} \tag{4.16}$$

on account of (2.11) and (2.12).

Since $\mathrm{Ai}''(x)=x\,\mathrm{Ai}(x)$, by (2.7) and (2.9)

$$|\,\mathrm{Ai}''(x)|\le|x|\,M(x)\le\frac{|x|^{3/4}}{\sqrt{\pi}} \tag{4.17}$$

if $x<-0.36605$. Note that θ_1 in (2.18) satisfies

$$\theta_1<\nu^{2/3}\zeta+\frac{B_0(\zeta)}{\nu^{4/3}(1+A_1(\zeta)/\nu^2)}<-0.36605;$$

cf. (4.14) and (4.15). Thus, we can apply (4.17) to (2.18). Furthermore, since

$$|\zeta|^{1/2}B_0(\zeta)\le 0.0109 \qquad \text{for } -\infty<\zeta<0$$

(see [10, p. 9], we have from (2.18) and (2.16)

$$|\varepsilon(\nu,\zeta)|\le\frac{1}{\sqrt{\pi}\nu^2|\nu^{2/3}\zeta|^{1/4}(1-1/225\nu^2)}\left[\frac{0.01324}{\nu}e^{0.22/\nu}+\frac{0.00006}{1-1/225\nu^2}\right].$$

Recall that $\nu^{2/3}\zeta<a_k+\rho'<0$; cf. (4.15). Hence, by (4.13)

$$\frac{1}{|\nu^{2/3}\zeta|^{1/4}}<\frac{1}{|a_k+\rho'|^{1/4}}=\frac{1}{|a_{k,0}|^{1/4}(1-0.01/(4k-1))^{1/4}},$$

and for $k\ge 3$

$$|\varepsilon(\nu,\zeta)|\le\frac{1.00023}{\sqrt{\pi}\nu^2|a_{k,0}|^{1/4}(1-1/225\nu^2)}\left[\frac{0.01324}{\nu}e^{0.22/\nu}+\frac{0.00006}{1-1/225\nu^2}\right]. \tag{4.18}$$

We rewrite (4.18) as

$$|\varepsilon(\nu,\zeta)| \le \frac{M_\nu}{\sqrt{\pi}\nu^2|a_{k,0}|^{1/4}}, \tag{4.19}$$

where

$$M_\nu = \begin{cases} 0.01663607 & \text{if } \nu \ge 1,\\ 7.459829728\times 10^{-3} & \text{if } \nu \ge 2,\\ 4.812652592\times 10^{-3} & \text{if } \nu \ge 3. \end{cases} \tag{4.20}$$

To estimate $|\operatorname{Ai}(a_k-\rho)|$ and $|\operatorname{Ai}(a_k+\rho')|$, we recall the asymptotic formula [12, p.394]

$$\operatorname{Ai}(-x) = \frac{1}{\sqrt{\pi}x^{1/4}}\left[\cos\left(\xi-\frac{\pi}{4}\right)+\varepsilon_1(\xi)\right], \tag{4.21}$$

where $\xi = \frac{2}{3}x^{3/2}$ and

$$|\varepsilon_1(\xi)| \le \frac{5}{72}\xi^{-1} + \frac{385}{10368}\xi^{-2}. \tag{4.22}$$

Let $x_1 = -(a_k-\rho)$ and $\xi_1 = \frac{2}{3}x_1^{3/2}$. Then by (4.13) and (4.10)

$$\begin{aligned}\xi_1 &= \frac{\pi}{4}(4k-1)\left(1+\frac{0.01}{4k-1}\right)^{3/2}\\ &= \left(k\pi-\frac{\pi}{4}\right)+\frac{\pi}{4}(0.01)\sum_{n=1}^{\infty}\binom{3/2}{n}\left(\frac{0.01}{4k-1}\right)^{n-1},\end{aligned}$$

which in turn gives

$$\begin{aligned}\left|\cos\left(\xi_1-\frac{\pi}{4}\right)\right| &= \left|\cos\left[k\pi-\frac{\pi}{2}+\frac{\pi}{4}(0.01)\sum_{n=1}^{\infty}\binom{3/2}{n}\left(\frac{0.01}{4k-1}\right)^{n-1}\right]\right|\\ &= \left|\sin\left[\frac{\pi}{4}(0.01)\sum_{n=1}^{\infty}\binom{3/2}{n}\left(\frac{0.01}{4k-1}\right)^{n-1}\right]\right|.\end{aligned}$$

Since

$$\frac{3}{2}+\frac{3}{8}x > \frac{(1+x)^{3/2}-1}{x} > \frac{3}{2}+\frac{3}{8}x-\frac{1}{16}x^2$$

for $x>0$, we have

$$\frac{3}{2}+\frac{3}{8}\left(\frac{0.01}{4k-1}\right) > \sum_{n=1}^{\infty}\binom{3/2}{n}\left(\frac{0.01}{4k-1}\right)^{n-1} > \frac{3}{2}-\frac{1}{16}\left(\frac{0.01}{4k-1}\right)^2.$$

Hence, for $k\ge 3$

$$0.011780972 < \frac{\pi}{4}(0.01)\sum_{n=1}^{\infty}\binom{3/2}{n}\left(\frac{0.01}{4k-1}\right)^{n-1} < 0.01178365$$

and

$$\left|\cos\left(\xi_1-\frac{\pi}{4}\right)\right| > 0.011780699. \tag{4.23}$$

From (4.22), we also have

$$
\begin{aligned}
|\varepsilon_1(\xi_1)| &\le \frac{5}{72}\frac{1}{(3\pi-\pi/4)} + \frac{385}{10368}\frac{1}{(3\pi-\pi/4)^2} \\
&= 8.535637057 \times 10^{-3}.
\end{aligned} \tag{4.24}
$$

A combination of (4.21), (4.23) and (4.24) yields

$$
\begin{aligned}
|\operatorname{Ai}(a_k-\rho)| &> \frac{1}{\sqrt{\pi}|a_{k,0}|^{1/4}(1+0.01/11)^{1/4}}\left|\left|\cos\left(\xi_1-\frac{\pi}{4}\right)\right| - |\varepsilon_1(\xi_1)|\right| \\
&> 3.244324849 \times 10^{-3}\frac{1}{\sqrt{\pi}|a_{k,0}|^{1/4}}
\end{aligned} \tag{4.25}
$$

for $k \ge 3$.

If $x_2 = -(a_k+\rho')$, then by (4.13) and (4.10)

$$
\begin{aligned}
\xi_2 = \frac{2}{3}x_2^{3/2} &= \left(k\pi-\frac{\pi}{4}\right)\left(1-\frac{0.01}{4k-1}\right)^{3/2} \\
&= k\pi - \frac{\pi}{4} - \frac{\pi}{4}(0.01)\left[\frac{3}{2} - \frac{3}{8}\frac{1}{(1+\theta_2)^{1/2}}\frac{0.01}{(4k-1)}\right],
\end{aligned}
$$

where $-0.01/(4k-1) < \theta_2 < 0$, and

$$
\left|\cos\left(\xi_2-\frac{\pi}{4}\right)\right| = \left|\sin\left\{\frac{\pi}{4}(0.01)\left[\frac{3}{2}-\frac{3}{8}\frac{1}{(1+\theta_2)^{1/2}}\frac{0.01}{(4k-1)}\right]\right\}\right|.
$$

For $k \ge 3$, this gives

$$
\left|\cos\left(\xi_2-\frac{\pi}{4}\right)\right| \ge 0.011778021\,. \tag{4.26}
$$

From (4.22), it also follows that for $k \ge 3$,

(4.27)

$$
\begin{aligned}
|\varepsilon_1(\xi_2)| &\le \frac{5}{72}\left(k\pi-\frac{\pi}{4}\right)^{-1}\left(1-\frac{0.01}{4k-1}\right)^{-\frac{3}{2}} + \frac{385}{10368}\left(k\pi-\frac{\pi}{4}\right)^{-2}\left(1-\frac{0.01}{4k-1}\right)^{-3} \\
&\le 8.547969921 \times 10^{-3}.
\end{aligned}
$$

Combining (4.21), (4.26) and (4.27), we obtain

$$
\begin{aligned}
|\operatorname{Ai}(a_k+\rho)| &> \frac{1}{\sqrt{\pi}}|a_{k,0}|^{-1/4}\left|\left|\cos\left(\xi_2-\frac{\pi}{4}\right)\right| - |\varepsilon_1(\xi_2)|\right| \\
&> 3.230051079 \times 10^{-3}\frac{1}{\sqrt{\pi}}|a_{k,0}|^{-1/4}.
\end{aligned} \tag{4.28}
$$

A comparsion of (4.19) with (4.25) and (4.28) yields

$$
E = \max|\varepsilon(\nu,\zeta)| < \min\big(|\operatorname{Ai}(a_k-\rho)|,\ |\operatorname{Ai}(a_k+\rho')|\big) \tag{4.29}
$$

for $\nu \ge 2$ and $k \ge 3$, thus establishing (4.2).

5. Estimation of $\min|\operatorname{Ai}'(x)|$

We first estimate $|\operatorname{Ai}'(a_k-\rho)|$ and $|\operatorname{Ai}'(a_k+\rho')|$. Corresponding to (4.21), we have from [12, p.392]

$$\operatorname{Ai}'(-x)=\pi^{-1/2}x^{1/4}\left[\sin\left(\xi-\frac{\pi}{4}\right)+\varepsilon_2(\xi)\right], \tag{5.1}$$

where $\xi=\frac{2}{3}x^{3/2}$ and

$$|\varepsilon_2(\xi)|\le\frac{7}{72}\xi^{-1}+\frac{455}{10368}\xi^{-2}+\frac{40415375}{644972544}\xi^{-4}. \tag{5.2}$$

Again let $x_1=-(a_k-\rho)$ and $\xi_1=\frac{2}{3}x_1^{3/2}$. By (4.13) and (4.10),

$$\xi_1=\frac{\pi}{4}(4k-1)\left(1+\frac{0.01}{4k-1}\right)^{3/2}$$

and

$$|\operatorname{Ai}'(a_k-\rho)|=\frac{1}{\sqrt{\pi}}|a_{k,0}|^{1/4}\left(1+\frac{0.01}{4k-1}\right)^{1/4}\left|\sin\left(\xi_1-\frac{\pi}{4}\right)+\varepsilon_2(\xi_1)\right|. \tag{5.3}$$

Note that

$$\left|\sin\left(\xi_1-\frac{\pi}{4}\right)\right|=\left|\cos\frac{\pi}{4}(4k-1)\left[\left(1+\frac{0.01}{4k-1}\right)^{3/2}-1\right]\right|.$$

Since $1<(1+x)^{3/2}<1+\frac{3}{2}x+\frac{3}{8}x^2$ for $x>0$, we have

$$0<\left(1+\frac{0.01}{4k-1}\right)^{3/2}-1<\frac{0.0150034091}{4k-1}$$

for all $k\ge 3$. Therefore,

$$\left|\sin\left(\xi_1-\frac{\pi}{4}\right)\right|>\left|\cos\frac{\pi}{4}(4k-1)\frac{0.0150034091}{4k-1}\right|>0.999930573. \tag{5.4}$$

From (5.2), it also follows that

$$|\varepsilon_2(\xi_1)|<0.011852593 \tag{5.5}$$

for $k\ge 3$. Here, use has been made of the fact that $4k-1\ge 11$. A combination of (5.3)–(5.5) gives

$$|\operatorname{Ai}'(a_k-\rho)|>\frac{0.98807798}{\sqrt{\pi}}|a_{k,0}|^{1/4}. \tag{5.6}$$

Similarly, let $x_2=-(a_k+\rho')$ and $\xi_2=\frac{3}{2}x^{3/2}$. Then

$$\xi_2=\frac{\pi}{4}(4k-1)\left(1-\frac{0.01}{4k-1}\right)^{3/2}$$

and

$$|\operatorname{Ai}'(a_k+\rho')|=\frac{1}{\sqrt{\pi}}|a_{k,0}|^{1/4}\left(1-\frac{0.01}{4k-1}\right)^{1/4}\left|\sin\left(\xi_2-\frac{\pi}{4}\right)+\varepsilon_2(\xi_2)\right|. \tag{5.7}$$

Since $1-\frac{3}{2}x<(1-x)^{3/2}<1$ for $0<x<1$, it follows that

$$0<1-\left(1-\frac{0.01}{4k-1}\right)^{3/2}<\frac{3}{2}\frac{0.01}{4k-1}$$

and

$$\left|\sin\Big(\xi_2 - \frac{\pi}{4}\Big)\right| > \cos\frac{\pi}{4}(4k-1)\frac{0.015}{4k-1} > 0.999930605\,. \tag{5.8}$$

For $k \geq 3$, we have from (5.2)

$$|\varepsilon_2(\xi_2)| < 0.01186962354\,. \tag{5.9}$$

Combining (5.7)–(5.9), we obtain

$$|\operatorname{Ai}'(a_k+\rho')| > \frac{0.987836345}{\sqrt{\pi}}|a_{k,0}|^{1/4}. \tag{5.10}$$

Set

$$m = \min\Big\{|\operatorname{Ai}'(x)| \ : a_k - \rho \leq x \leq a_k + \rho'\Big\}.$$

Since a'_k and a'_{k+1} are two consecutive zeros of $\operatorname{Ai}'(x)$, and since a_k is a critical point of $\operatorname{Ai}'(x)$ in $[a'_{k+1}, a'_k]$, the minimum value m is attained at the endpoints $a_k - \rho$ and $a_k + \rho'$. From (5.6) and (5.10), it follows that

$$\begin{aligned} m &= \min\Big(|\operatorname{Ai}'(a_k-\rho)|,\ |\operatorname{Ai}'(a_k+\rho')|\Big) \\ &> \frac{0.987836345}{\sqrt{\pi}}|a_{k,0}|^{1/4}. \end{aligned} \tag{5.11}$$

A comparision of (4.19) and (5.11) shows that $m > E$. Hence, by Hethcote's theorem, we have from (3.3) and (3.4)

$$|\eta_k| \leq \frac{E}{m} < \frac{\lambda_k}{\nu^2} \tag{5.12}$$

for $k \geq 3$, where

$$\lambda_k = \frac{M_\nu}{0.987836345}|a_{k,0}|^{-1/2}. \tag{5.13}$$

6. The function $x'''(\zeta)$ in (3.9)

Let $x(\zeta)$ be the inverse function of $\zeta(x)$ defined by (2.2)–(2.3). In [4], Hethcote has shown that $x''(\zeta)$ is a non-decreasing function of ζ for $\zeta \leq 0$ or $x \geq 1$. Here, we are concerned with the function $x'''(\zeta)$; cf., equation (3.9). The graph of this function is depicted in Figure 1 below, and it is shown in [7] that $x'''(\zeta)$ is a decreasing function for $-0.526 < \zeta < 0$ and that

$$\frac{6}{700} \leq x'''(\zeta) \leq 0.03474367\,.$$

In this paper, we shall prove a more global result by using a more direct method. The precise statement of the result is given in the following lemma.

Lemma 1. *In the interval* $(-\infty, 0)$, $x'''(\zeta)$ *has one and only one critical point, and it is a maximum.*

Proof. From [4, p.85], we have

(6.1)

$$x'''(\zeta) = \frac{x}{4(-\zeta)^{3/2}(x^2-1)^{7/2}}\Big\{(x^2-1)^3 + 6(-\zeta)^{\frac{3}{2}}(x^2-1)^{\frac{3}{2}} - 4(3x^2+1)(-\zeta)^3\Big\}.$$

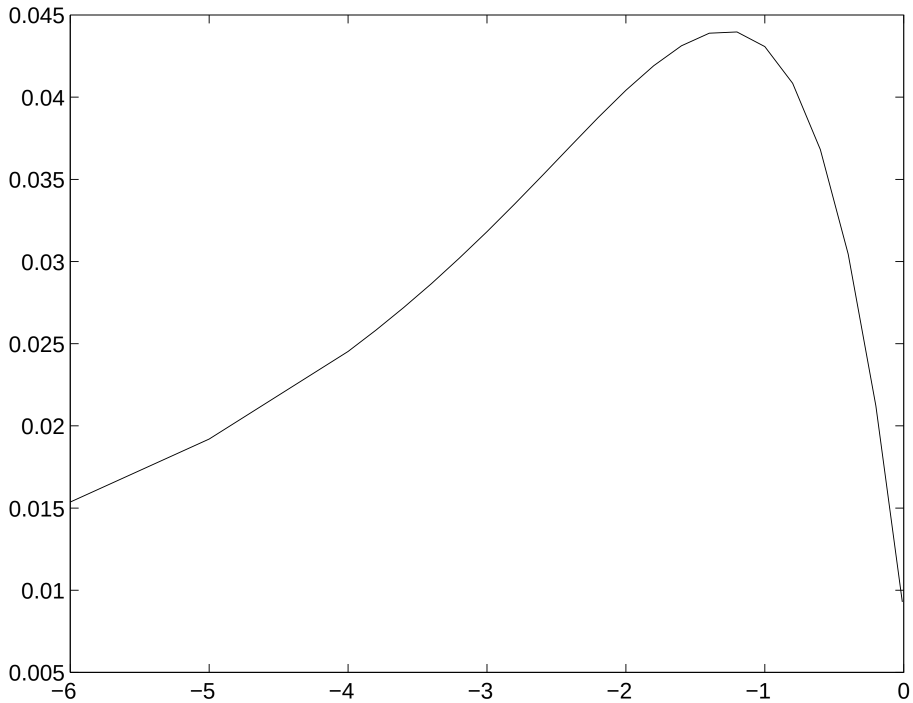


FIGURE 1. Graph of $x'''(\zeta)$

For convenience, we put

$$u \equiv \sqrt{x^2-1} \qquad \text{and} \qquad w \equiv u - \arctan u. \tag{6.2}$$

Direct computation gives

$$x^{(4)}(\zeta) = \frac{27x}{8(-\zeta)^{5/2}u^{10}}\left\{\frac{u^9}{9} + \frac{u^6 w}{9} + (6u^5+8u^3)w^2 - (12u^4+39u^2+28)w^3\right\}; \tag{6.3}$$

see [7, (A.5)]. Let

$$G(u) \equiv \frac{u^9}{9} + \frac{u^6 w}{9} + (6u^5+8u^3)w^2 - (12u^4+39u^2+28)w^3. \tag{6.4}$$

To prove Lemma 1, we need to show that $x^{(4)}(\zeta)$ has one and only one zero ζ_0 in $(-\infty,0)$, $x^{(4)}(\zeta) > 0$ in $(-\infty,\zeta_0)$, and $x^{(4)}(\zeta) < 0$ in $(\zeta_0, 0)$. For $\zeta \in (-\infty, 0)$, i.e., $x > 1$, it is clear that the factor outside the curly brackets in (6.3) is positive. Hence, $x^{(4)}(\zeta)$ and $G[u(\zeta)]$ have the same zeros and the same signs in $(-\infty,0)$. Note that $u = 0$ corresponds to $\zeta = 0$, $u = \infty$ corresponds to $\zeta = -\infty$, and that the interval $0 < u < \infty$ corresponds to the interval $-\infty < \zeta < 0$. As a consequence, we only need to prove that $G(u)$ has one and only one zero u_0 in $0 < u < \infty$, $G(u) < 0$ in $(0, u_0)$, and $G(u) > 0$ in (u_0,∞). To achieve this, we consider the derivatives of $G(u)$. Since $dw/du = u^2/(1+u^2)$, after a certain number of differentiations, the derivatives of $G(u)$ will simply become rational functions of u. From (6.4), it is easily seen that $G^{(5)}(u)$ does not contain w^3. Note that

$$\frac{d^2w}{du^2} = \frac{2u}{(1+u^2)^2}.$$

Since the degree of the numerator of d^2w/du^2 is smaller than that of the numerator of dw/du, for simplicity, we differentiate $G(u)$ six times. The result is

$$\begin{aligned} G^{(6)}(u) = \frac{16}{3}\Big\{&175u^3 + 2205u^5 + 7623u^7 + 8375u^9 + 8085u^{11} \\ &+ 5220u^{13} + 1260u^{15} - w(u)\Big[525 + 6930u^2 \\ &+ 26613u^4 + 50406u^6 + 48375u^8 + 19350u^{10} \\ &+ 3225u^{12}\Big] - w^2(u)\Big[567u + 5454u^3 \\ &+ 7047u^5\Big]\Big\}\Big/(1+u^2)^6. \end{aligned}$$

Since the factor $16/3(1+u^2)^6$ is always positive and does not effect the zeros or the signs of $G^{(6)}(u)$, we need consider only the function

$$\begin{aligned} H(u) = \Big[&175u^3 + 2205u^5 + 7623u^7 + 8375u^9 + 8085u^{11} \\ &+ 5220u^{13} + 1260u^{15}\Big] - w(u)\Big[525 + 6930u^2 + 26613u^4 \\ &+ 50406u^6 + 48375u^8 + 19350u^{10} + 3225u^{12}\Big] \\ &- w^2(u)\Big[567u + 5454u^3 + 7047u^5\Big]. \end{aligned}$$

Differentiating $H(u)$ six times eliminates $w^2(u)$, and we obtain

$$\begin{aligned} H^{(6)}(u) = 1728\Big\{&-152352u^3 - 945447u^5 + 714600u^7 + 16982800u^9 \\ &+ 55789200u^{11} + 93786450u^{13} + 93249800u^{15} \\ &+ 55568100u^{17} + 18433800u^{19} + 2627625u^{21} \\ &- w(u)\Big[22423 + 730161u^2 + 5481888u^4 \\ &+ 20403500u^6 + 44537250u^8 + 61113750u^{10} \\ &+ 53642500u^{12} + 29347500u^{14} + 9142875u^{16} \\ &+ 1241625u^{18}\Big]\Big\}\Big/(1+u^2)^6. \end{aligned} \tag{6.5}$$

Let $K(u)$ denote the quantity inside the curly brackets in (6.5). Then $K(u)$ and $H^{(6)}(u)$ differ only by a positive factor and, hence, have the same zeros and the same signs. The term with the highest power of u containing $w(u)$ is $u^{18}w(u)$. After 19 differentiations, $K(u)$ will become a rational function whose denominator is positive. Using analysis, we can determine accurately the zeros of $K^{(19)}(u)$ and the regions where $K^{(19)}(u)$ is positive or negative. By pure luck, it turns out that all we need is to differentiate $K(u)$ eight times. The result is

$$\begin{aligned}K^{(8)}(u) = 2304\Big\{&980848627u + 87953839791u^3\\ &+ 2020254381801u^5 + 21069256637762u^7\\ &+ 126228135835875u^9 + 488255296930125u^{11}\\ &+ 1300787764107625u^{13} + 2476116094147500u^{15}\\ &+ 3431552631278625u^{17} + 3479130358559625u^{19}\\ &+ 2558273260417875u^{21} + 1330056483005250u^{23}\\ &+ 464092433429625u^{25} + 97597189564875u^{27}\\ &+ 9357169696875u^{29} + \arctan u\Big[779401875\\ &+ 54362293125u^2 + 871518033750u^4\\ &+ 6650914436250u^6 + 30158142853125u^8\\ &+ 90042909691875u^{10} + 186876398572500u^{12}\\ &+ 277273313047500u^{14} + 297153403798125u^{16}\\ &+ 228820451146875u^{18} + 123705072933750u^{20}\\ &+ 44638119776250u^{22} + 9665547766875u^{24}\\ &+ 950792968125u^{26}\Big]\Big\}\Big/(1+u^2)^8.\end{aligned}$$

Since $K^{(8)}(u) > 0$ for $0 < u < \infty$, $K^{(7)}(u)$ is monotonically increasing in $(0,\infty)$. Simple calculation shows

(6.6) $$\lim_{u\to 0^+} K^{(7)}(u) = -4888130112, \qquad \lim_{u\to\infty} K^{(7)}(u) = \infty.$$

Therefore, $K^{(7)}(u)$ has one and only one zero u_7^K in $(0,\infty)$ such that $K^{(7)}(u) < 0$ in $(0, u_7^K)$ and $K^{(7)}(u) > 0$ in (u_7^K, ∞). This, in turn, implies that $K^{(6)}(u)$ is strictly decreasing in $(0, u_7^K)$ and strictly increasing in (u_7^K, ∞). Since

(6.7) $$\lim_{u\to 0^+} K^{(6)}(u) = 0 \qquad \text{and} \qquad \lim_{u\to\infty} K^{(6)}(u) = \infty,$$

it follows that $K^{(6)}(u)$ has a unique zero u_6^K which lies in (u_7^K,∞), i.e., $0 < u_7^K < u_6^K < \infty$. Furthermore, $K^{(6)}(u)$ is negative in $(0, u_6^K)$ and positive in (u_6^K, ∞). By the same argument, we obtain the conclusions in Table 1.

TABLE 1. Behavior of $K(u)$

j	$K^{(j)}(u)$	$K^{(j)}(0^+)$	$K^{(j)}(\infty)$	$\searrow$	$\nearrow$	zero	$-$	$+$
7	$K^{(7)}(u)$	-48881	∞		$(0,\infty)$	$u_7^K > 0$	$(0,u_7^K)$	(u_7^K,∞)
6	$K^{(6)}(u)$	0	∞	$(0,u_7^K)$	(u_7^K,∞)	$u_6^K > u_7^K$	$(0,u_6^K)$	(u_6^K,∞)
5	$K^{(5)}(u)$	-14212	∞	$(0,u_6^K)$	(u_6^K,∞)	$u_5^K > u_6^K$	$(0,u_5^K)$	(u_5^K,∞)
4	$K^{(4)}(u)$	0	∞	$(0,u_5^K)$	(u_5^K,∞)	$u_4^K > u_5^K$	$(0,u_4^K)$	(u_4^K,∞)
3	$K^{(3)}(u)$	-95895	∞	$(0,u_4^K)$	(u_4^K,∞)	$u_3^K > u_4^K$	$(0,u_3^K)$	(u_3^K,∞)
2	$K^{(2)}(u)$	0	∞	$(0,u_3^K)$	(u_3^K,∞)	$u_2^K > u_3^K$	$(0,u_2^K)$	(u_2^K,∞)
1	$K^{(1)}(u)$	0	∞	$(0,u_2^K)$	(u_2^K,∞)	$u_1^K > u_2^K$	$(0,u_1^K)$	(u_1^K,∞)
0	$K^{(0)}(u)$	0	∞	$(0,u_1^K)$	(u_1^K,∞)	$u_0^K > u_1^K$	$(0,u_0^K)$	(u_0^K,∞)

Since $H^{(6)}(u)$ and $K(u)$ have the same zeros and the same signs, $H^{(6)}(u) < 0$ in $(0, u_0^K)$, $H^{(6)}(u_0^K) = 0$, and $H^{(6)}(u) > 0$ in (u_0^K, ∞). Using the same argument, we come to the conclusions in Table 2.

TABLE 2. Behavior of $H(u)$

j	$H^{(j)}(u)$	$H^{(j)}(0^+)$	$H^{(j)}(\infty)$	$\searrow$	$\nearrow$	zero	$-$	$+$
6	$H^{(6)}(u)$	0	$+\infty$			$u_6^H = u_0^K$	$(0, u_6^H)$	(u_6^H, ∞)
5	$H^{(5)}(u)$	0	$+\infty$	$(0, u_6^H)$	(u_6^H, ∞)	$u_5^H > u_6^H$	$(0, u_5^H)$	(u_5^H, ∞)
4	$H^{(4)}(u)$	0	$+\infty$	$(0, u_5^H)$	(u_5^H, ∞)	$u_4^H > u_5^H$	$(0, u_4^H)$	(u_4^H, ∞)
3	$H^{(3)}(u)$	0	$+\infty$	$(0, u_4^H)$	(u_4^H, ∞)	$u_3^H > u_4^H$	$(0, u_3^H)$	(u_3^H, ∞)
2	$H^{(2)}(u)$	0	$+\infty$	$(0, u_3^H)$	(u_3^H, ∞)	$u_2^H > u_3^H$	$(0, u_2^H)$	(u_2^H, ∞)
1	$H^{(1)}(u)$	0	$+\infty$	$(0, u_2^H)$	(u_2^H, ∞)	$u_1^H > u_2^H$	$(0, u_1^H)$	(u_1^H, ∞)
0	$H^{(0)}(u)$	0	$+\infty$	$(0, u_1^H)$	(u_1^H, ∞)	$u_0^H > u_1^H$	$(0, u_0^H)$	(u_0^H, ∞)

Since $G^{(6)}(u)$ and $H(u)$ have the same zeros and the same signs, we conclude that $G^{(6)}(u_0^H) = 0$, $G^{(6)}(u) < 0$ in $(0, u_0^H)$, and $G^{(6)}(u) > 0$ in (u_0^H, ∞). Put $u_6^G \equiv u_0^H$. Then, again by the same argument, we obtain the results in Table 3.

TABLE 3. Behavior of $G(u)$

j	$G^{(j)}(u)$	$G^{(j)}(0^+)$	$G^{(j)}(\infty)$	$\searrow$	$\nearrow$	zero	$-$	$+$
6	$G^{(6)}(u)$	0	$+\infty$			$u_6^G = u_0^H$	$(0, u_6^G)$	(u_6^G, ∞)
5	$G^{(5)}(u)$	0	$+\infty$	$(0, u_6^G)$	(u_6^G, ∞)	$u_5^G > u_6^G$	$(0, u_5^G)$	(u_5^G, ∞)
4	$G^{(4)}(u)$	0	$+\infty$	$(0, u_5^G)$	(u_5^G, ∞)	$u_4^G > u_5^G$	$(0, u_4^G)$	(u_4^G, ∞)
3	$G^{(3)}(u)$	0	$+\infty$	$(0, u_4^G)$	(u_4^G, ∞)	$u_3^G > u_4^G$	$(0, u_3^G)$	(u_3^G, ∞)
2	$G^{(2)}(u)$	0	$+\infty$	$(0, u_3^G)$	(u_3^G, ∞)	$u_2^G > u_3^G$	$(0, u_2^G)$	(u_2^G, ∞)
1	$G^{(1)}(u)$	0	$+\infty$	$(0, u_2^G)$	(u_2^G, ∞)	$u_1^G > u_2^G$	$(0, u_1^G)$	(u_1^G, ∞)

From (6.4) and the properties of $G'(u)$ in Table 3, it follows that $G(0) = 0$, $G(\infty) = \infty$, and $G(u)$ has one and only one zero, say at $u = u_0$, in the interval (u_1^G, ∞). Furthermore, $G(u)$ is negative in $(0, u_0)$ and positive in (u_0, ∞). This completes the proof of Lemma 1. $\square$

For the convenience of the reader, a computer print-out of the MAPLE commands is provided in Table 4.

Since $G(u)$ has only one zero in $(0, \infty)$, the value of its zero u_0 can be accurately calculated using Newton's method. The result is

$$u_0 = 2.086469208\,. \tag{6.8}$$

The corresponding zero ζ_0 of $x^{(4)}(\zeta)$ is

$$\zeta_0 = -1.277497301\,. \tag{6.9}$$

Numerical computation also gives

$$\max_{-\infty<\zeta<0} x'''(\zeta) = x'''(\zeta_0) = 0.0440358004\,. \tag{6.10}$$

Since $x'''(-\infty) = 0$ and $x'''(0^-) = 3/350$, we obtain

$$0 < x'''(\zeta) < 0.045, \qquad -\infty < \zeta < 0. \tag{6.11}$$

Table 4. Maple commands

```
>G:=u^9/9+u^6*w(u)/9+(6*u^5+8*u^3)*w(u)^2-(12*u^4+39*u^2+28)*w(u)^3;
>W:=u-arctan(u);
>w1:=diff(W,u);
>g6:=diff(G,u$6);
>G6:=simplify(subs(diff(w(u),u)=w1,g6));
>H:=sort(collect(G6*(1+u^2)^6/(16/3),w(u)));
>h6:=diff(H,u$6);
>H6:=simplify(subs(diff(w(u),u)=w1,h6));
>K:=sort(collect(H6*(1+u^2)^6/1728,w(u)));
>k8:=diff(K,u$8);
>K8:=simplify(subs(w(u)=W,k8));
>x3:=x*((x^2-1)^3+6*(-Z)^(3/2)*(x^2-1)^(3/2)-4*(3*x^2+1)*(-Z)^3)/(4*(-Z)^(3
>/2)*(x^2-1)^(7/2));
>z:=-((3/2)*(u-arctan(u)))^(2/3);
>X3:=subs(Z=z,x=(u^2+1)^(1/2),x3);
>limit(X3,u=0,right);
>limit(X3,u=infinity);
>g:=subs(w(u)=W,G);
>u0:=fsolve(g=0,u=1..5);
>`max{X3}`:=evalf(subs(u=u0,X3));
>z0:=evalf(subs(u=u0,z));
```

From (6.11), it is clear that 0 is a lower bound for $x'''(\zeta)$ in $(-\infty, 0)$. But, in order to prove Lorch's conjecture, we need a more precise estimate for $x'''(\zeta)$ when $-\infty < \zeta < \zeta_0$.

Returning to (3.7), we note that the quantity θ in $x'''(\theta)$ lies in between ζ_k and 0. From (4.14), we have

$$\left[a_k - \rho - \frac{B_0(\zeta_k)}{\nu^{4/3}(1 + A_1(\zeta_k)/\nu^2)}\right]\nu^{-2/3} < \zeta_k < (a_k + \rho')\nu^{-2/3} < 0. \tag{6.12}$$

What we need is a lower bound of $x'''(\zeta)$ in $(\zeta_k^*, 0)$, where

$$\zeta_k^* = \left(a_k - \rho - \frac{B_0(\zeta_k)}{\nu^{4/3}(1 + A_1(\zeta_k)/\nu^2)}\right)\nu^{-2/3}. \tag{6.13}$$

By Lemma 1, there is a unique $\zeta_1 \in (-\infty, \zeta_0)$ such that

$$x'''(\zeta_1) = x'''(0) = \frac{3}{350};$$

see Figure 1. From (6.1), it is found by numerical computation that

$$\zeta_1 = -9.253401702\,. \tag{6.14}$$

Lower bounds of $x'''(\zeta)$ in $(\zeta_k^*, 0)$ can be obtained by considering two separate cases: (i) $\zeta_k^* \geq \zeta_1$ and (ii) $\zeta_k^* < \zeta_1$.

Lemma 2. *Let ζ_k^* be defined as in* (6.13). *For all $\nu \geq 2$ and $k \geq 3$,*

$$\zeta_k^* \geq \frac{1.0014}{\nu^{2/3}} a_{k,0}. \tag{6.15}$$

Furthermore, if $\zeta_k^* \geq \zeta_1$, then

$$x'''(\zeta) \geq \frac{3}{350} \qquad \textit{for } \zeta_k^* < \zeta < 0, \tag{6.16}$$

and if $\zeta_k^* < \zeta_1$, *then*

$$x'''(\zeta) \geq \frac{0.142\nu}{|a_{k,0}|^{3/2}} \qquad \text{for } \zeta_k^* < \zeta < \zeta_1. \tag{6.17}$$

Proof. We first recall (2.16) and the estimate $|\zeta|^{1/2} B_0(\zeta) \leq 0.0109$ for $\zeta \in (-\infty, 0)$. By (6.12) and (4.13), we have

$$\frac{B_0(\zeta_k)}{\nu^{4/3}(1 + A_1(\zeta_k)/\nu^2)} \leq \frac{0.0109}{\nu |a_{k,0}|^{1/2}(1 - 0.01/(4k-1))^{1/2}(1 - 1/225\nu^2)}.$$

The definition of ζ_k^* in (6.13) then gives

$$\begin{aligned}\zeta_k^* &\geq \left[a_{k,0}\left(1 + \frac{0.01}{11}\right) - \frac{0.0109}{2|a_{k,0}|^{1/2}(1 - 0.01/11)^{1/2}(1 - 1/(225 \times 4))}\right]\nu^{-2/3} \\ &\geq a_{k,0}\nu^{-2/3}\left[\left(1 + \frac{0.01}{11}\right) + \frac{0.0109}{2|a_{k,0}|^{3/2}(1 - 0.01/11)^{1/2}(1 - 1/(225 \times 4))}\right]\end{aligned}$$

for $\nu \geq 2$ and $k \geq 3$. Here again use has been made of (4.13). In view of (4.10), the result in (6.15) now follows.

The lower bound in (6.16) is obtained immediately from Lemma 1. By that lemma, we also have $x'''(\zeta) > x'''(\zeta_k^*)$ for $\zeta_k^* < \zeta < \zeta_1$. Hence, it suffices to show that (6.17) holds when $\zeta = \zeta_k^*$. Let u_k be defined by

$$\frac{2}{3}\left(-\zeta_k^*\right)^{3/2} = u_k - \arctan u_k\,. \tag{6.18}$$

Since $\zeta_k^* < \zeta_1$ and $u_k > 0$, by (6.14) we have $u_k > 1$. Furthermore, since

$$\frac{1}{u} + \arctan u > \frac{\pi}{2} \qquad \text{for } 1 < u < \infty, \tag{6.19}$$

it follows that

$$\frac{2}{3}\left(-\zeta_k^*\right)^{3/2} < \frac{2}{3}\left(-\zeta_k^*\right)^{3/2} + \frac{\pi}{2} - \frac{1}{u_k} < u_k < \frac{2}{3}\left(-\zeta_k^*\right)^{3/2} + \frac{\pi}{2}\,. \tag{6.20}$$

Expressing (6.1) in terms of u and applying (6.19), we obtain

$$\begin{aligned}x'''(\zeta_k^*) &= \frac{\sqrt{u_k^2 + 1}}{4(-\zeta_k^*)^{3/2} u_k^7}\left[u_k^6 + 6(-\zeta_k^*)^{3/2} u_k^3 - 4(3u_k^2 + 4)(-\zeta_k^*)^3\right] \\ &> \frac{\frac{2}{3}(-\zeta_k^*)^{3/2}\left[1 + \frac{9}{4}(-\zeta_k^*)^{-3}\right]^{1/2}}{4(-\zeta_k^*)^{3/2}\left(\frac{2}{3}\right)^7(-\zeta_k^*)^{21/2}\left[1 + \frac{3\pi}{4}(-\zeta_k^*)^{-3/2}\right]^7}\left(\frac{2}{3}\right)^6\left(-\zeta_k^*\right)^9 F(\zeta_k^*),\end{aligned} \tag{6.21}$$

where

$$\begin{aligned}F(\zeta_k^*) = \frac{1}{(\frac{2}{3})^6(-\zeta_k^*)^9}\Bigg\{&\left[\frac{2}{3}(-\zeta_k^*)^{3/2} + m\right]^6 + 6(-\zeta_k^*)^{3/2}\left[\frac{2}{3}(-\zeta_k^*)^{3/2} + m\right]^3 \\ &- \left(12\left[\frac{2}{3}(-\zeta_k^*)^{3/2} + \frac{\pi}{2}\right]^2 + 16\right)\left(-\zeta_k^*\right)^3\Bigg\}\end{aligned}$$

and $m = \pi/2 - 1/u_k$. Since $\zeta_k^* < \zeta_1$, by (6.14) and (6.19) we have $m > 1.517507135$. Applying this to (6.21) gives

$$\begin{aligned}F(\zeta_k^*) > 1 &+ 13.65756422(-\zeta_k^*)^{-3/2} + 37.22044179(-\zeta_k^*)^{-3} \\ &+ 87.8878569(-\zeta_k^*)^{-9/2} + 197.9527061(-\zeta_k^*)^{-6} \\ &+ 605.4892582(-\zeta_k^*)^{-15/2} + 139.1015613(-\zeta_k^*)^{-9} > 1.\end{aligned}$$

Therefore, by virtue of (6.21),

$$x'''(\zeta_k^*) > \frac{\left[1+\frac{9}{4}(-\zeta_k^*)^{-3}\right]^{1/2}}{4(-\zeta_k^*)^{3/2}\left[1+\frac{3\pi}{4}(-\zeta_k^*)^{-3/2}\right]^7}$$
$$> \frac{1}{4(-\zeta_k^*)^{3/2}\left[1+\frac{3\pi}{4}(-\zeta_1)^{-3/2}\right]^7}.$$

Replacing ζ_1 by its value in (6.14) yields

$$x'''(\zeta_k^*) > 0.1424(-\zeta_k^*)^{-3/2}. \tag{6.22}$$

The desired result (6.17) now follows from (6.15) and (6.22). □

From (6.15), it is evident that $\zeta_k^* > \zeta_1$ if

$$\nu^{2/3} \geq \frac{1.0014(-a_{k,0})}{-\zeta_1}.$$

By (4.10) and (6.14), this inequality holds as long as

$$\nu \geq 0.042(4k-1), \qquad k \geq 3. \tag{6.23}$$

Therefore, we may use (6.16) whenever ν and k satisfy (6.23). If (6.23) fails to hold, then either $\zeta_k^* \geq \zeta_1$ or $\zeta_k^* < \zeta_1$. In either case, we can use (6.17) since

$$\frac{3}{350} > \frac{0.142\nu}{|a_{k,0}|^{3/2}} \qquad \text{when } \nu < 0.042(4k-1).$$

7. Proof of (1.2): The upper bound

Since Lorch and Uberti [9] already have established the upper bound in (1.2) when $0 < \nu \leq 10$ and $k = 1, 2, 3$, and since Lang and Wong [7] have already proved its validity when $10 \leq \nu < \infty$ and $k = 1, 2$, it suffices to consider just the case $k \geq 3$. In view of the remark at the end of the previous section, our discussion may be divided into four distinct cases:

(i) $2 \leq \nu < \infty$, $k \geq 3$, $\nu \geq 0.042(4k-1)$,
(ii) $2 \leq \nu < \infty$, $k \geq 3$, $\nu < 0.042(4k-1)$,
(iii) $0 < \nu < 2$, $k \geq 12$,
(iv) $0 < \nu < 2$, $k = 4, 5, 6, 7, 8, 9, 10, 11$.

Case (i): $2 \leq \nu < \infty$, $k \geq 3$, $\nu \geq 0.042(4k-1)$. From (3.5) and (5.12), we have

$$-\frac{B_0(\zeta_k)}{(1+A_1(\zeta_k)/\nu^2)} - \frac{\lambda_k}{\nu^{2/3}} \leq \delta_k \leq -\frac{B_0(\zeta_k)}{(1+A_1(\zeta_k)/\nu^2)} + \frac{\lambda_k}{\nu^{2/3}}. \tag{7.1}$$

A combination of (5.13), (4.10) and (4.20) gives

$$\lambda_k \leq \frac{0.0075516859}{\left[\frac{3\pi}{8}(4k-1)\right]^{1/3}} \tag{7.2}$$

for $\nu \geq 2$. Since $B_0(\zeta)$ is increasing and $A_1(\zeta)$ is decreasing in $(-\infty, 0)$, it follows that

$$0 < B_0(\zeta_1) < B_0(\zeta_k) < B_0(0) = \frac{2^{4/3}}{140}$$

and

$$-\frac{1}{225} = A_1(0) < A_1(\zeta_k) < 0$$

for $\zeta_1 < \zeta_k^* < \zeta_k$; see [7, (2.20)–(2.21)] and also equation (2.16) in § 2. The inequalities in (7.1) then give

$$-\frac{B_0(0)}{1+A_1(0)/\nu^2} - \frac{\lambda_k}{\nu^{2/3}} < \delta_k < -B_0(\zeta_1) + \frac{\lambda_k}{\nu^{2/3}}.$$

By (7.2), we have

$$0 < \lambda_k \leq 0.0032150370 \tag{7.3}$$

for $k \geq 3$. Since

$$B_0(0) = \frac{2^{4/3}}{140} \qquad \text{and} \qquad B_0(\zeta_1) = 0.0008171994,$$

we obtain

$$-0.0200442396 < \delta_k < 0.0012081470\,. \tag{7.4}$$

In the following, we shall use $\delta_{k,l}$ and $\delta_{k,r}$ to denote the lower and upper bounds of δ_k in (7.4), respectively.

For convenience, let us put

$$\beta_1 = -2^{-1/3}\frac{\delta_k}{a_k^3}, \qquad \beta_2 = \frac{3}{20}2^{1/3}\delta_k\left(\frac{2}{a_k^2} + \frac{\delta_k}{a_k^3}\nu^{-4/3}\right)\nu^{-2/3},$$

and

$$\beta_3 = \frac{x'''(\theta)}{6}\left(1 + \frac{\delta_k \nu^{-4/3}}{a_k}\right)^3,$$

so that (3.9) becomes

$$\beta = a_k^3\big(\beta_1 + \beta_2 + \beta_3\big). \tag{7.5}$$

From (4.3), (4.4) and (4.5), we have

$$a_{k,0}\left\{1 + \frac{0.130}{\left[\frac{3\pi}{8}(4k - 1.051)\right]^2}\right\} \leq a_k \leq a_{k,0}.$$

In view of (4.10), the above lower bound can be replaced by

$$a_{k,l} = -1.116331775(4k-1)^{2/3},$$

and we obtain

$$a_{k,l} < a_k \leq a_{k,0} \leq a_{3,0} < 0 \tag{7.6}$$

for $k \geq 3$. This, coupled with (7.4), yields

$$-\frac{2^{-1/3}\delta_{k,l}}{a_{3,0}^3} < \beta_1 < -\frac{2^{-1/3}\delta_{k,r}}{a_{3,0}^3}$$

since $\delta_{k,l}$ is negative. Computation gives

$$-9.473241352 \times 10^{-5} < \beta_1 < 5.709903868 \times 10^{-6} \tag{7.7}$$

on account of (4.10) and (7.4). Again, since $\delta_{k,l}$ is negative, we also have

$$\delta_{k,l}2^{-4/3} < \delta_k \nu^{-4/3} < \delta_{k,r}2^{-4/3}$$

and

$$0 < 2 + \frac{\delta_{k,r}2^{-4/3}}{a_{3,0}} < 2 + \frac{\delta_k \nu^{-4/3}}{a_k} < 2 + \frac{\delta_{k,l}2^{-4/3}}{a_{3,0}}.$$

Therefore,

$$0 < \frac{1}{a_k^2}\left(2 + \frac{\delta_k \nu^{-4/3}}{a_k}\right)\nu^{-2/3} < \frac{1}{a_{3,0}^2}\left(2 + \frac{\delta_{k,l} 2^{-4/3}}{a_{3,0}}\right)2^{-2/3}$$

and

$$\frac{3}{20}2^{1/3}\frac{\delta_{k,l}}{a_{3,0}^2}\left(2 + \frac{\delta_{k,l}2^{-4/3}}{a_{3,0}}\right)2^{-2/3} < \beta_2 < \frac{3}{20}2^{1/3}\frac{\delta_{k,r}}{a_{3,0}^2}\left(2 + \frac{\delta_{k,l}2^{-4/3}}{a_{3,0}}\right)2^{-2/3}.$$

Computation gives

$$-0.0001569093082 < \beta_2 < 0.9457555572 \times 10^{-5}. \tag{7.8}$$

By virtue of the remark at the end of § 6, in the present case, we have $\zeta_k^* > \zeta_1$. Hence,

$$\frac{3}{350} \le x'''(\zeta) \le 0.045$$

on account of (6.11) and (6.16), which in turn gives

$$\frac{1}{700}\left(1 + \frac{\delta_{k,r}2^{-4/3}}{a_{3,0}}\right)^3 < \beta_3 < \frac{0.045}{6}\left(1 + \frac{\delta_{k,l}2^{-4/3}}{a_{3,0}}\right)^3$$

and

$$0.001428199024 < \beta_3 < 0.007532486957\,. \tag{7.9}$$

A combination of (7.7)–(7.9) yields

$$0.007547654416 a_k^3 < \beta < 0.001176557302 a_k^3\,. \tag{7.10}$$

Therefore, β is negative, and it follows from (3.8) that the second inequality in (1.2) holds.

Case (ii): $2 \le \nu < \infty$, $k \ge 3$, $\nu < 0.042(4k-1)$. Since $B_0(\zeta)$ is increasing in $(-\infty, 0)$ and $\zeta_k < 0$, we have

$$0 < B_0(\zeta_k) < B_0(0) = \frac{2^{4/3}}{140}. \tag{7.11}$$

Coupling this with (7.1) gives

$$-\frac{B_0(0)}{1 + A_1(0)/\nu^2} - \frac{\lambda_k}{\nu^{2/3}} < \delta_k < \frac{\lambda_k}{\nu^{2/3}}. \tag{7.12}$$

Using the estimates of λ_k in (7.3), we get

$$-0.0200442396 < \delta_k < 0.002025346396 \tag{7.13}$$

for $k \ge 3$ and $\nu \ge 2$. Let $\delta_{k,l}^*$ and $\delta_{k,r}^*$ denote the lower and upper bound of δ_k in (7.13), respectively, so that

$$\delta_{k,l}^* < \delta_k < \delta_{k,r}^*\,.$$

By (6.11) and the remark following Lemma 2,

$$\frac{0.142\nu}{|a_{k,0}|^{3/2}} < x'''(\theta) < 0.045\,. \tag{7.14}$$

Put

$$\beta_1^* = \frac{2^{-1/3}\delta_k}{|a_k|^{3/2}\nu}, \qquad \beta_2^* = \frac{3}{20}2^{1/3}\frac{\delta_k}{|a_k|^{1/2}}\left(2 + \frac{\delta_k \nu^{-4/3}}{a_k}\right)\nu^{-5/3},$$

and

$$\beta_3^* = \frac{x'''(\theta)}{6}\frac{|a_k|^{3/2}}{\nu}\left(1+\frac{\delta_k\nu^{-4/3}}{a_k}\right)^3.$$

Equation (3.9) now can be written as

$$\beta = -|a_k|^{3/2}\nu\big(\beta_1^* + \beta_2^* + \beta_3^*\big)\,. \tag{7.15}$$

By (7.6) and (7.13),

$$\frac{2^{-1/3}\delta_{k,l}^*}{|a_{3,0}|^{3/2}2} < \beta_1^* < \frac{2^{-1/3}\delta_{k,r}^*}{|a_{3,0}|^{3/2}2}$$

for $\nu \geq 2$. Numerical computation gives

$$-0.0006138219745 < \beta_1^* < 0.00006202291276\,. \tag{7.16}$$

As to β_2^*, the argument is similar to that for β_2 in Case (i), and we have

$$\frac{3}{20}2^{1/3}\frac{\delta_{k,l}^*}{|a_{3,0}|^{1/2}}\left(2+\frac{\delta_{k,l}^*2^{-4/3}}{a_{3,0}}\right)2^{-5/3} < \beta_2^* < \frac{3}{20}2^{1/3}\frac{\delta_{k,r}^*}{|a_{3,0}|^{1/2}}\left(2+\frac{\delta_{k,l}^*2^{-4/3}}{a_{3,0}}\right)2^{-5/3}.$$

Direct computation yields

$$-0.001016699330 < \beta_2^* < 0.000102731177\,. \tag{7.17}$$

By using (7.14), we get

$$\frac{1}{6}\frac{0.142\nu}{|a_{k,0}|^{3/2}}\frac{|a_k|^{3/2}}{\nu}\left(1+\frac{\delta_{k,r}^*2^{-4/3}}{a_{3,0}}\right)^3 < \beta_3^* < \frac{0.045}{6}\frac{|a_k|^{3/2}}{\nu}\left(1+\frac{\delta_{k,l}^*2^{-4/3}}{a_{3,0}}\right)^3.$$

From (7.6), it follows that

$$\frac{1}{6}\frac{0.142\nu}{|a_{k,0}|^{3/2}}\frac{|a_{k,0}|^{3/2}}{\nu}\left(1+\frac{\delta_{k,r}^*2^{-4/3}}{a_{3,0}}\right)^3 < \beta_3^* < \frac{0.045}{6}\frac{|a_{k,l}|^{3/2}}{\nu}\left(1+\frac{\delta_{k,l}^*2^{-4/3}}{a_{3,0}}\right)^3.$$

By direct computation, we obtain

$$0.02365632468 < \beta_3^* < 0.01776880880k - 0.004442202199\,. \tag{7.18}$$

A combination of (7.16)–(7.18) gives

(7.19)

$$-(0.01776880880k - 0.004277448109)|a_k|^{3/2}\nu < \beta < -0.02202580338|a_k|^{3/2}\nu;$$

thus, β is negative.

Case (iii): $0 < \nu < 2$, $k \geq 12$. In [4], Hethcote has shown that

$$\begin{aligned}&\left|j_{\nu,k} - \left(k+\frac{\nu}{2}-\frac{1}{4}\right)\pi - \frac{4\nu^2-1}{8\pi(k+\nu/2-1/4)}\right| \\ &\qquad \leq \frac{0.0246|4\nu^2-1||4\nu^2-9| + 0.0288|4\nu^2-1|}{(k+\nu/2-1/2)^2} + \frac{0.00016|4\nu^2-1|^3}{(k+\nu/2-1/4)^3}\end{aligned} \tag{7.20}$$

for $k \geq \frac{1}{2} - \frac{\nu}{2} + \max\{3.2, 0.76|4\nu^2-1|\}$. Since $0 < \nu \leq 2$, it follows that $\frac{1}{2} - \frac{\nu}{2} + \max\{3.2, 0.76|4\nu^2-1|\} \leq 11.9$, i.e., (7.20) automatically holds in the present case. Furthermore, since the right-hand side of (7.20) is easily shown to be less than 0.024, (7.20) gives

$$j_{\nu,k} < \left(k + \frac{\nu}{2} - \frac{1}{4}\right)\pi + \frac{4\nu^2 - 1}{8\pi(k + \nu/2 - 1/4)} + 0.024 \tag{7.21}$$

for $k \geq 12$ and $0 < \nu < 2$. Recall that $a_k < a_{k,0} < 0$, cf. (7.6). Thus,

(7.22)

$$\begin{aligned}\nu - \left(\frac{\nu}{2}\right)^{1/3} a_k + \frac{3}{20}\left(\frac{2}{\nu}\right)^{1/3} a_k^2 &> \nu - \left(\frac{\nu}{2}\right)^{1/3} a_{k,0} + \frac{3}{20}\left(\frac{2}{\nu}\right)^{1/3} a_{k,0}^2 \\ &= \nu + \left(\frac{\nu}{2}\right)^{1/3}\left(\frac{3\pi}{8}\right)^{2/3}(4k-1)^{2/3} + \frac{3}{20}\left(\frac{2}{\nu}\right)^{1/3}\left(\frac{3\pi}{8}\right)^{4/3}(4k-1)^{4/3}.\end{aligned}$$

Denote the right-hand sides of (7.21) and (7.22) by $Z(\nu, k)$ and $U(\nu, k)$, respectively. To establish the second inequality in (1.2), it suffices to show that $U(\nu, k) > Z(\nu, k)$ for (ν, k) in $D = \{(\nu, k) :\ 0 < \nu \leq 2,\ k \geq 12\}$. Put

$$F(\nu, k) = U(\nu, k) - Z(\nu, k)\,. \tag{7.23}$$

Clearly,

$$\begin{aligned}\frac{\partial F}{\partial k} = {} & \left(\frac{\nu}{2}\right)^{1/3}\left(\frac{3\pi}{8}\right)^{2/3}\frac{8}{3}(4k-1)^{-1/3} + \frac{\nu^2}{2\pi(k+\nu/2-1/4)^2} \\ & + \frac{4}{5}\left(\frac{2}{\nu}\right)^{1/3}\left(\frac{3\pi}{8}\right)^{4/3}(4k-1)^{1/3}\left\{1 - \frac{5}{4}\left(\frac{\nu}{2}\right)^{1/3}\left(\frac{8}{3\pi}\right)^{4/3}\right. \\ & \left. \times (4k-1)^{-1/3}\left[\pi + \frac{1}{8\pi(k+\nu/2-1/4)^2}\right]\right\}.\end{aligned}$$

The quantity inside the curly brackets is greater than 0.1253675546, and hence $\partial F/\partial k > 0$ for $(\nu, k) \in D$. Similarly,

$$\begin{aligned}\frac{\partial F}{\partial \nu} = {} & -\frac{2^{1/3}}{20}\left(\frac{3\pi}{8}\right)^{4/3}(4k-1)^{4/3}\frac{1}{\nu^{4/3}}\left\{1 - \frac{20}{2^{2/3}}\left(\frac{8}{3\pi}\right)^{2/3}\frac{1}{3}\frac{\nu^{2/3}}{(4k-1)^{2/3}}\right. \\ & \left. - \left[1 + \frac{4\nu^2-1}{16\pi(k+\nu/2-1/4)^2}\right]\frac{20}{2^{1/3}}\left(\frac{8}{3\pi}\right)^{4/3}\frac{\nu^{4/3}}{(4k-1)^{4/3}}\right\} - \frac{\pi}{2} \\ & - \frac{\nu}{\pi(k+\nu/2-1/4)},\end{aligned}$$

and the quantity inside the curly brackets is greater than 0.3511510032. Hence, $\partial F/\partial \nu < 0$ in D. From these, we conclude that the minimum of $F(\nu, k)$ in D occurs at $k = 12$ and $\nu = 2$, and that its value is $F(2, 12) = 8.057746222$. Therefore, $F(\nu, k) > 0$ and $U(\nu, k) > Z(\nu, k)$.

Case (iv): $0 < \nu < 2$, $k =$ **4, 5, 6, 7, 8, 9, 10, 11.** By differentiating with respect to ν, it can be shown easily that for each fixed k, $U(\nu, k)$ has a minimum $\nu_0(k)$ in $(0, \infty)$, and its value is $\nu_0(k) = 0.06942430471(4k-1)$. For $8 \leq k \leq 11$, we have $\nu_0(k) > 2$. Table 5 lists the values of $U(2, k)$ and $j_{2,k}$ for $k = 8, 9, 10, 11$. Since $j_{\nu,k}$ is increasing in ν (see [12, p.246]), and $U(\nu, k)$ is decreasing in $0 < \nu \leq 2$, it follows that

$$j_{\nu,k} < j_{2,k} < U(2, k) < U(\nu, k) < \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} + \frac{3}{20}2^{1/3}a_k^2\nu^{-1/3}$$

for $0 < \nu \leq 2$ and $8 \leq k \leq 11$. When $4 \leq k \leq 7$, we have $\nu_0(k) < 2$.

TABLE 5

k	8	9	10	11
$U(2,k)$	31.18297036	35.30293870	39.51233654	43.80718300
$j_{2,k}$	27.42057355	30.56920450	33.71651951	36.86285651

Values of $\nu_0(k)$, $U(\nu_0(k),k)$ and $j_{2,k}$ for $k = 4$, 5, 6 and 7 are given in Table 6. Since $j_{\nu,k}$ is increasing in ν and the minimal value of $U(\nu,k)$ is $U(\nu_0(k),k)$, we infer from Table 6 that

$$j_{\nu,k} < j_{2,k} < U(\nu_0(k),k) < U(\nu,k)$$

for $4 \le k \le 7$ and $0 < \nu \le 2$.

TABLE 6

k	4	5	6	7
$\nu_0(k)$	1.041364571	1.319061787	1.596759004	1.874456225
$U(\nu_0(k),k)$	15.08160077	19.10336098	23.12512117	27.14688139
$j_{2,k}$	14.79595178	17.95981950	21.11699705	24.27011231

This completes the proof of the second inequality in (1.2).

8. PROOF OF (1.2): THE LOWER BOUND

In a similar manner, we shall show in the following that the first two terms in the asymptotic expansion (1.1) form a lower bound for $j_{\nu,k}$, i.e.,

$$j_{\nu,k} > \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} \tag{8.1}$$

for all $\nu > 0$ and all $k = 1, 2, \ldots$. When $k = 1$, this has already been established in [4] for $\nu > 1$ and extended by Laforgia and Muldoon in [6] for all $\nu > 0$. When $k = 2$, a proof of (8.1) can be found in [7] and [9]. What remain to be proved are the three cases:

(i) $\nu \ge 2$ and $k \ge 3$,
(ii) $0 < \nu \le 2$ and $k \ge 12$,
(iii) $0 < \nu \le 2$ and $3 \le k \le 11$.

Case (i): $\nu \ge 2$ and $k \ge 3$. Taking the first two terms in the Maclaurin expansion of $x(\zeta)$, we have

$$\begin{aligned} x_k = x(\zeta_k) &= 1 - 2^{-1/3}\zeta_k + \frac{1}{2!}x''(\theta^*)\zeta_k^2 \\ &= 1 - 2^{-1/3}a_k\nu^{-2/3} - 2^{-1/3}\delta_k\nu^{-2} + \frac{1}{2!}x''(\theta^*)\big(a_k + \delta_k\nu^{-4/3}\big)^2\nu^{-4/3}, \end{aligned}$$

where $\zeta_k < \theta^* < 0$; cf. (3.7). Since $j_{\nu,k} = \nu x_k$, we may write

$$j_{\nu,k} = \nu - \frac{a_k}{2^{1/3}}\nu^{1/3} + \alpha\nu^{-1/3} \tag{8.2}$$

with

$$\alpha = -2^{-1/3}\delta_k\nu^{-2/3} + \frac{1}{2}x''(\theta^*)\big(a_k + \delta_k\nu^{-4/3}\big)^2. \tag{8.3}$$

Recall from (6.11) that $x'''(\zeta) > 0$ for $\zeta \in (-\infty, 0)$. Here, $x''(\zeta)$ is an increasing function in $(-\infty, 0)$. Also, it is easily computed that

$$\lim_{\zeta \to 0^-} x''(\zeta) = \frac{3}{10} 2^{1/3} = 0.377976315 \tag{8.4}$$

and

$$\lim_{\zeta \to -\infty} x''(\zeta) = 0\,. \tag{8.5}$$

Let ζ_k^* be defined as in (6.13). Since $\zeta_k^* < \zeta_k < 0$, by monotonicity,

$$x''(\zeta_k^*) < x''(\theta^*) < x''(0). \tag{8.6}$$

To estimate $x''(\zeta_k^*)$, we let $u = \sqrt{x^2 - 1}$. From [4], we have

$$\begin{aligned} x''(\zeta) &= \frac{\sqrt{u^2+1}}{2(-\zeta)^{1/2} u^4} \Big[u^3 - 3(u - \arctan u) \Big] \\ &= \frac{\sqrt{1 + 1/u^2}}{2(-\zeta)^{1/2}} \left[1 - \frac{3(u - \arctan u)}{u^3} \right]. \end{aligned} \tag{8.7}$$

It can be shown that the function inside the square brackets in (8.7) is increasing for $u > 1$, and that it attains the value $\frac{1}{2}$ at $u = \bar{u} = 1.374077402$. Hence, for $u > \bar{u}$, the value of this function is greater than $\frac{1}{2}$. From (8.7), it then follows that

$$x''(\zeta_k^*) > \frac{1}{4(-\zeta_k^*)^{1/2}} \tag{8.8}$$

for $\zeta_k^* < \bar{\zeta}$, where

$$\bar{\zeta} = -0.7492884996$$

is the value of ζ corresponding to $\bar{u}$; see (2.3). Coupling (6.15) and (8.8) gives

$$x''(\zeta_k^*) > \frac{0.24 \nu^{1/3}}{|a_{k,0}|^{1/2}}\,. \tag{8.9}$$

If $\zeta_k^* \geq \bar{\zeta}$, then by monotonicity,

$$x''(\zeta_k^*) \geq x''(\bar{\zeta}) = 0.3571984309\,. \tag{8.10}$$

We now are ready to derive bounds for the quantity α in (8.3). First, we consider the case $\nu \geq 1.8202(4k-1)$. On account of (6.15), this implies $\zeta_k^* > \bar{\zeta}$. Hence, we can use the result in (8.10). Write α in the form

$$\alpha = a_k^2 \left[-\frac{\delta_k}{a_k^2 2^{1/3} \nu^{2/3}} + \frac{1}{2} x''(\theta^*) \left(1 + \frac{\delta_k}{a_k \nu^{4/3}} \right)^2 \right].$$

A combination of (7.4), (7.6), (8.3) and (8.10) gives

$$0.178548334 a_k^2 < \alpha < 0.189862763 a_k^2\,. \tag{8.11}$$

If $\nu < 1.8202(4k-1)$, there are still two possibilities, namely, $\zeta_k^* < \bar{\zeta}$ and $\zeta_k^* \geq \bar{\zeta}$. In the case $\zeta_k^* < \bar{\zeta}$, we can use (8.9). In the case $\zeta_k^* \geq \bar{\zeta}$, since

$$\frac{0.24 \nu^{1/3}}{|a_{k,0}|^{1/2}} < \frac{0.24[1.8202(4k-1)]^{1/3}}{[3\pi(4k-1)/8]^{1/3}} < 0.2774537282,$$

we also can use (8.9) by virtue of (8.10). Write α as

$$\alpha = |a_k|^{3/2}\left[-\frac{\delta_k}{|a_k|^{3/2}2^{1/3}\nu^{2/3}} + \frac{x''(\theta^*)}{2}|a_k|^{1/2}\left(1+\frac{\delta_k}{a_k\nu^{4/3}}\right)^2\right].$$

Then, from (7.13), (7.6), (8.6) and (8.9), it follows that

$$0.151068333 < \alpha|a_k|^{-3/2} < \left[0.0007733672265 + 0.2002546693(4k-1)^{1/3}\right]. \tag{8.12}$$

(Note that $\delta_{k,l} = \delta^*_{k,l}$ and $\delta_{k,r} < \delta^*_{k,r}$.) Coupling (8.11) and (8.12) together establishes (8.1) for $\nu \geq 2$ and $k \geq 3$.

Case (ii): $0 < \nu \leq 2$ and $k \geq 12$. By the argument leading to (7.21), we have from (7.20)

$$j_{\nu,k} > \left(k + \frac{\nu}{2} - \frac{1}{4}\right)\pi + \frac{4\nu^2 - 1}{8\pi(k+\nu/2-1/4)} - 0.024 \tag{8.13}$$

for $0 < \nu \leq 2$ and $k \geq 12$. From (4.4) and (4.5), we also have

$$a_k > a_{k,0}\left\{1 + \frac{0.130}{[3\pi(4k-1.051)/8]^2}\right\}.$$

When $k \geq 3$, the last inequality gives

$$a_k > 1.0008a_{k,0}.$$

Hence,

$$\begin{aligned}\nu - \frac{a_k}{2^{1/3}}\nu^{1/3} &< \nu - \frac{1.0008}{2^{1/3}}a_{k,0}\nu^{1/3} \\ &= \nu + \frac{1.0008}{2^{1/3}}\left(\frac{3\pi}{8}\right)^{2/3}(4k-1)^{2/3}\nu^{1/3}\end{aligned} \tag{8.14}$$

for $k \geq 3$. Let $J(\nu,k)$ and $V(\nu,k)$ denote, respectively, the right-hand side of the inequalities (8.13) and (8.14). To establish (8.1) in the present case, it suffices to show that the function

$$Q(\nu,k) \equiv J(\nu,k) - V(\nu,k)$$

is positive in $D = \{(\nu,k):\ 0 < \nu \leq 2 \text{ and } k \geq 12\}$. By straightforward differentiation, we obtain

$$\begin{aligned}\frac{\partial Q}{\partial k} = \pi&\left[1 - \frac{4\nu^2}{8\pi^2(k+\nu/2-1/4)^2} - \frac{1.0008}{2^{1/3}}\left(\frac{3\pi}{8}\right)^{2/3}\frac{8}{3}\frac{\nu^{1/3}}{\pi(4k-1)^{1/3}}\right] \\ &+ \frac{1}{8\pi(k+\nu/2-1/4)^2}.\end{aligned}$$

The quantity inside the square brackets is greater than 0.7359566840 for $(\nu,k) \in D$. Hence, $\partial Q/\partial k > 0$ in D. Similarly, we have

$$\begin{aligned}\frac{\partial Q}{\partial \nu} = -\frac{(4k-1)^{2/3}}{\nu^{2/3}}&\left\{\frac{1.0008}{2^{1/3}}\left(\frac{3\pi}{8}\right)^{2/3}\frac{1}{3} - \frac{\nu^{2/3}}{(4k-1)^{2/3}}\left[\frac{\pi}{2} + \frac{\nu}{\pi(k+\nu/2-1/4)}\right.\right. \\ &\left.\left. + \frac{1}{16\pi(k+\nu/2-1/4)^2}\right]\right\} - 1 - \frac{4\nu^2}{16\pi(k+\nu/2-1/4)^2}.\end{aligned}$$

The quantity inside the curly brackets is greater than 0.09726998. Hence, $\partial Q/\partial \nu < 0$ in D. From these, one easily deduces that $Q(\nu,k)$ attains its minimum in D at

$\nu = 2$ and $k = 12$. Since $Q(2, 12) = 23.53915816$, we conclude that $Q(\nu, k) > 0$ in D, i.e.,

$$j_{\nu,k} > J(\nu, k) > V(\nu, k) > \nu - \frac{a_k}{2^{1/3}}\nu^{1/3}$$

for $0 < \nu \leq 2$ and $k \geq 12$.

Case (iii): $0 < \nu \leq 2$ **and** $3 \leq k \leq 11$**.** Since for each fixed k both $V(\nu, k)$ and $j_{\nu,k}$ are increasing in $0 < \nu < \infty$, it is evident from the values of $j_{0,k}$ and $V(2, k)$ in Table 7 that

$$j_{\nu,k} > j_{0,k} > V(2, k) > V(\nu, k)$$

for $0 < \nu \leq 2$ and $3 \leq k \leq 11$. The inequality in (8.1) now follows from (8.14).

TABLE 7

k	$j_{0,k}$	$V(2,k)$
3	8.653727913	7.521577602
4	11.79153444	8.789882045
5	14.93091771	9.948840651
6	18.07106397	11.02859033
7	21.21163663	12.04717345
8	24.35247153	13.01647161
9	27.49347913	13.94483312
10	30.63460647	14.83840667
11	33.77582021	15.70188635

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